THE GROUPS OF POINTS ON ABELIAN VARIETIES OVER FINITE FIELDS

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Abstract. Let A be an abelian variety with commutative endomorphism algebra over a finite field k. The k-isogeny class of A is uniquely determined by a Weil polynomial f_A without multiple roots. We give a classification of the groups of k-rational points on varieties from this class in terms of Newton polygons of $f_A(1-t)$.

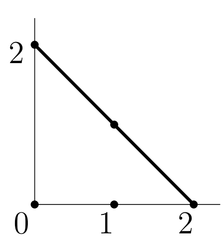
1. Introduction

Let A be an abelian variety of dimension g over a finite field $k = \mathbb{F}_q$, and let $A(k)$ be the group of k-rational points on A. Tsfasman [Ts85] classified all possible groups $A(k)$, where A is an elliptic curve (see the English exposition in [TsVN07, 3.3.15]). Later the same result was independently proved in [Ru87] and [Vo88] using [Sch87]. Xing obtained a similar classification when A is a supersingular simple surface [Xi94] and [Xi96]. In this paper such a description is obtained for the groups of k-rational points on abelian varieties with commutative endomorphism algebra.

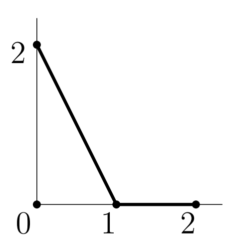
For an abelian group H we denote by H_ℓ the ℓ-primary component of H. Let $A(k) = \oplus_\ell A(k)_\ell$. We associate to $A(k)_\ell$ a polygon of special type.

Definition 1. Let $0 \le m_1 \le m_2 \le \cdots \le m_r$ be nonnegative integers, and let $H = \oplus_{i=1}^r \mathbb{Z}/\ell^{m_i}\mathbb{Z}$ be an abelian group of order ℓ^m. The *Hodge polygon* $\mathrm{Hp}_\ell(H, r)$ *of a group* H is the convex polygon with vertices $(i, \sum_{j=1}^{r-i} m_j)$ for $0 \le i \le r$. It has $(0, m)$ and $(r, 0)$ as its endpoints, and its slopes are $-m_r, \dots, -m_1$.

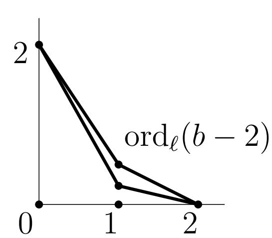
Note that Hodge polygon of a group depends on a choice of r, but the zero slope of $\mathrm{Hp}_\ell(A(k)_\ell, r)$ is not interesting, and we can use the most convenient choice of r. Mostly we need the case $r = 2g$, and write $\mathrm{Hp}_\ell(H) = \mathrm{Hp}_\ell(H, 2g)$ (here we assume that H can be generated by $2g$ elements). For example, let $g = 1$, and $H = \mathbb{Z}/\ell\mathbb{Z} \oplus \mathbb{Z}/\ell\mathbb{Z}$, then $\mathrm{Hp}_\ell(H, 2)$ is a straight line (see Picture 1). Let $H = \mathbb{Z}/\ell^2\mathbb{Z}$ be cyclic, then $\mathrm{Hp}_\ell(H, 2)$ has a zero slope (see Picture 2).



Pic. 1



Pic. 2



Pic. 3

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Note also that isomorphism class of H depends only on $\mathrm{Hp}_\ell(H)$. For a polynomial $P \in \mathbb{Z}[t]$ we denote by $\mathrm{Np}_\ell(P)$ the Newton polygon of P with respect to ℓ (see Section 3 for the precise definition). The aim of this paper is to prove the following theorem.

Theorem 1. *Let A be an abelian variety over a finite field with Weil polynomial f_A (see Section 2 for the definition of Weil polynomial). Suppose f_A has no multiple roots (i.e. endomorphism algebra $\mathrm{End}^\circ(A)$ is commutative). Let G be an abelian group of order $f_A(1)$. Then G is a group of points on some variety in the isogeny class of A if and only if $\mathrm{Np}_\ell(f_A(1-t))$ lies on or above $\mathrm{Hp}_\ell(G_\ell)$ for any prime number ℓ.*

As an example, we prove the following corollary of Theorem 1. Originally it was proved in [Ts85]. Later the same result was independently proved in [Ru87] and [Vo88] using [Sch87].

Corollary 1. *Let $N = 1 - b + q$ be the order of $B(k)$ for an elliptic curve B. Then $G = B(k)$ satisfies the following conditions.*

(1) *If $b \neq \pm 2\sqrt{q}$, then $G \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z}$, where $N = n_1 n_2$, and n_1 divides $b-2$ and n_2.*
(2) *If $b = \pm 2\sqrt{q}$, then $G \cong (\mathbb{Z}/n_1\mathbb{Z})^2$, and $N = n_1^2$.*

If a finite commutative group G satisfies (1) or (2), then there exists an elliptic curve B' isogenous to B such that $B'(k) \cong G$.

Proof. If $b \neq \pm 2\sqrt{q}$, then f_B has no multiple roots. Fix a prime ℓ. Let $B(k)_\ell \cong \mathbb{Z}/\ell^{m_1}\mathbb{Z} \oplus \mathbb{Z}/\ell^{m_2}\mathbb{Z}$, and let $m_1 \leq m_2$. Since $f_B(1-t) = t^2 + (b-2)t + (1-b+q)$, we have by Theorem 1 that $m_1 \leq \mathrm{ord}_\ell(b-2)$ (see Picture 3). Equivalently, ℓ^{m_1} divides $b-2$, and the first case follows. The second case is obvious since F acts on T_ℓ as multiplication by $b/2 = \pm\sqrt{q}$. Conversely, if G satisfies (1), then the existence of an elliptic curve B' with $B'(k) \cong G$ follows from Theorem 1 combined with the inequality $m_1 \leq \mathrm{ord}_\ell(b-2)$. □

Remark 1. Originally the Tsfasman theorem relies on the Waterhouse classification of isogeny classes of elliptic curves [Wa69] and has 6 cases. We make the statement shorter, but it becomes a little different. For example, for a supersingular curve B with commutative endomorphism algebra it is not immediately clear that $B(k)$ is cyclic modulo 2-torsion.

Corollary 1 allows us to construct an example of an isogeny class with the following properties:

(1) The Weil polynomial f has multiple roots;
(2) not any abelian group G of order $f(1)$ such that $\mathrm{Np}_\ell(f(1-t))$ lies on or above $\mathrm{Hp}_\ell(G_\ell)$ for any prime number ℓ is a group of points on some variety from the isogeny class.

Indeed, take $f(t) = (t-3)^2$. Then by Corollary 1 for any A from the isogeny class $A(\mathbb{F}_9) \cong (\mathbb{Z}/2\mathbb{Z})^2$. It follows that the isogeny class does not contain an elliptic curve, whose group of points is cyclic.

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2. Preliminaries

Throughout this paper k is a finite field $\mathbb{F}_q$ of characteristic p. Let A and B be abelian varieties over k. It is well known that the group $\mathrm{Hom}(A,B)$ of k-homomorphisms from A to B is finitely generated and torsionfree. We use the following notation: $\mathrm{Hom}^\circ(A,B) = \mathrm{Hom}(A,B) \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\mathrm{End}^\circ(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. The algebra $\mathrm{End}^\circ(A)$ contains the Frobenius endomorphism F, and its center is equal to $\mathbb{Q}[F]$. Thus $\mathrm{End}^\circ(A)$ is commutative if and only if $\mathrm{End}^\circ(A) = \mathbb{Q}[F]$.

Let A be an abelian variety of dimension g over k, and let $\bar{k}$ be an algebraic closure of k. For a natural number m denote by A_m the kernel of multiplication by m in $A(\bar{k})$. Let $A[m]$ be the group subscheme of A, which is the kernel of multiplication by m. By definition $A_m = A[m](\bar{k})$. Let $T_\ell(A) = \varprojlim A_{\ell^r}$ be the Tate module, and $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ be the corresponding vector space over $\mathbb{Q}_\ell$. If $\ell \neq p$, then $T_\ell(A)$ is a free $\mathbb{Z}_\ell$-module of rank $2g$. The Frobenius endomorphism F of A acts on the Tate module by a semisimple linear operator, which we also denote by $F : T_\ell(A) \to T_\ell(A)$. The characteristic polynomial

$$f_A(t) = \det(t - F|T_\ell(A))$$

is called a *Weil polynomial of* A. It is a monic polynomial of degree $2g$ with rational integer coefficients independent of the choice of prime $\ell \neq p$. It is well known that for isogenous varieties A and B we have $f_A(t) = f_B(t)$. Moreover, Tate proved that the isogeny class of abelian variety is determined by its characteristic polynomial, that is $f_A(t) = f_B(t)$ implies that A is isogenous to B. The polynomial f_A has no multiple roots if and only if the endomorphism algebra $\mathrm{End}^\circ(A)$ is commutative (see [WM69]).

If $\ell = p$, then $T_p(A)$ is called a *physical Tate module*. In this case, $f_A(t) = f_1(t)f_2(t)$, where $f_1, f_2 \in \mathbb{Z}_p[t]$, and $f_1(t) = \det(t - F|T_p(A))$. Moreover $d = \deg f_1 \leq g$, and $f_2(t) \equiv t^{2g-d} \mod p$ (see [De78]).

We say that $\varphi : B \to A$ is an *ℓ-isogeny*, if degree of φ is a power of ℓ. The following lemma is well known, but we prove it here for the sake of completeness.

Lemma 1. *If $\varphi : B \to A$ is an isogeny then $T_\ell(\varphi) : T_\ell(B) \to T_\ell(A)$ is a $\mathbb{Z}_\ell$-linear embedding commuting with the action of the Frobenius endomorphisms and if T denotes its image then*

$$F(T) \subset T \quad \text{and} \quad T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell. \tag{1}$$

Conversely, if $T \subset T_\ell(A)$ is a $\mathbb{Z}_\ell$-submodule such that (1) *holds, then there exists an abelian variety B over k, and an ℓ-isogeny $\varphi : B \to A$ such that $T_\ell(\varphi)$ induces an isomorphism $T_\ell(B) \cong T$.*

Proof. The first part is obvious. The second part can be proved as follows. First, (1) implies that there exists $k \in \mathbb{Z}$ such that $\ell^k T_\ell(A) \subset T$. Further, note that $T_\ell(A)/\ell^k T_\ell(A) \cong A_{\ell^k}$. Hence the group $T/\ell^k T_\ell(A)$ can be considered as a subgroup in $A_{\ell^k} \subset A(\bar{k})$. Moreover, since $F(T) \subset T$ it follows that $T/\ell^k T_\ell(A)$ is invariant under the action of the Frobenius, and thus defines a certain group subscheme G of A. If $\ell \neq p$, we define $B = A/G$. If $\ell = p$, there is the canonical decomposition $A[p^k] = G_r \oplus G_l$, where G_r is reduced and $G_r(\bar{k}) = A_{p^k}$, and $G_l(\bar{k}) = 0$ [De78]. In this case, we define $B = A/(G \oplus G_l)$. It is clear that B is defined over k, and $A \cong B/G'$, where G' is reduced and $G'(\bar{k}) = T_\ell(A)/T$. This gives a desired isogeny $\varphi : B \to A$. □

3. The groups of points

Let $Q(t) = \sum_i Q_i t^i$ be a polynomial of degree d over $\mathbb{Q}_\ell$, and let $Q(0) = Q_0 \neq 0$. Take the lower convex hull of the points $(i, \mathrm{ord}_\ell(Q_i))$ for $0 \leq i \leq d$ in $\mathbb{R}^2$. The boundary of this region without vertical lines is called *the Newton polygon* $\mathrm{Np}_\ell(Q)$ *of* Q. Its vertices have integer coefficients, and $(0, \mathrm{ord}_\ell(Q_0))$ and $(d, \mathrm{ord}_\ell(Q_d))$ are its endpoints.

Let E be an injective endomorphism of $T_\ell(A)$, and let $H = T_\ell(A)/ET_\ell(A)$ be its cokernel, which is a finite ℓ-group. Define the *Hodge polygon of the endomorphism* E as the Hodge polygon $\mathrm{Hp}_\ell(H, \mathrm{rk}\, T_\ell(A))$ of the group H. We need the following simple result.

Proposition 1. *The Hodge polygon $\mathrm{Hp}_\ell(A(k)_\ell, \mathrm{rk}\, T_\ell(A))$ is equal to the Hodge polygon of $1 - F$.*

Proof. Since $A(k)_\ell$ is finite, there exists a positive integer N such that $A(k)_\ell \subset A_{\ell^N}$. Clearly, $A(k)_\ell = \ker(1-F : A_{\ell^N} \to A_{\ell^N})$. Apply $1-F$ to the short exact sequence:

$$0 \to T_\ell(A) \xrightarrow{\ell^N} T_\ell(A) \to A_{\ell^N} \to 0.$$

By snake lemma we get:

$$0 = \ker(1-F : T_\ell(A) \to T_\ell(A)) \to A(k)_\ell \to T_\ell(A)/(1-F)T_\ell(A) \xrightarrow{0} T_\ell(A)/(1-F)T_\ell(A).$$

Thus $A(k)_\ell \cong T_\ell(A)/(1-F)T_\ell(A)$, and the proposition follows. $\square$

We see that the group $A(k)_\ell$ depends only on the action of F on the Tate module $T_\ell(A)$. In particular, order of $A(k)$ is $f_A(1)$. Thus we reduced our task to the following linear algebra problem. We are given a $\mathbb{Q}_\ell$-vector space V of finite positive dimension d and an invertible linear operator $E : V \to V$, whose characteristic polynomial $f(t) = \det(E - t|V)$ lies in $\mathbb{Z}_\ell[t]$. We want to describe all finite commutative groups (up to isomorphism) of the form T/ET, where T is an arbitrary E-invariant $\mathbb{Z}_\ell$-lattice of rank d in V. We solve this problem under the additional assumption that f has no multiple roots.

The next statement establishes a connection between the Hodge polygon of an endomorphism and the Newton polygon of its characteristic polynomial.

Theorem 2. [Ke09, 4.3.8][BO, 8.40] *Let E be an injective endomorphism of a free $\mathbb{Z}_\ell$-module of finite rank. Let $f(t) = \det(E - t|V)$ be its characteristic polynomial. Then $\mathrm{Np}_\ell(f)$ lies on or above the Hodge polygon of E, and these polygons have same endpoints.*

Recall that if $\ell = p$, then $f_A(t) = f_1(t)f_2(t)$, and $f_2(t) \equiv t^{2g-d} \mod p$. Thus the only slope of $\mathrm{Np}_p(f_2(1-t))$ is zero, and $\mathrm{Np}_p(f_A(1-t))$ equals to $\mathrm{Np}_p(f_1(1-t))$ up to this zero slope.

Corollary 2. *Let A be an abelian variety A with Weil polynomial f_A. Then $\mathrm{Np}_\ell(f_A(1-t))$ lies on or above $\mathrm{Hp}_\ell(A(k)_\ell)$, and these polygons have same endpoints $(0, \mathrm{ord}_\ell(f_A(1))$ and $(2g, 0)$.*

Now we are ready to solve our linear algebra problem.

Theorem 3. *Let V be a $\mathbb{Q}_\ell$-vector space of positive finite dimension d, and let $E : V \to V$ be an invertible linear operator such that the characteristic polynomial $f(t) = \det(E - t|V)$ lies in $\mathbb{Z}_\ell[t]$. Suppose f has no multiple roots. Let G be an abelian group of order ℓ^m, where $m = \mathrm{ord}_\ell(f(0))$. If $\mathrm{Np}_\ell(f)$ lies on or above $\mathrm{Hp}_\ell(G, d)$, then there exists an E-invariant $\mathbb{Z}_\ell$-lattice T of rank d in V such that $T/ET \cong G$.*

Corollary 3. *We keep the notation of theorem 3. Suppose f has no multiple roots. Then the group G is isomorphic to T/ET for some E-invariant $\mathbb{Z}_\ell$-lattice T of rank d in V if and only if $\mathrm{Np}_\ell(f)$ lies on or above $\mathrm{Hp}_\ell(G, d)$, and these polygons have same endpoints $(0, \mathrm{ord}_\ell(f(0))$ and $(d, 0)$.*

Proof of theorem 3. Since f has no multiple roots, there is an isomorphism of $\mathbb{Q}_\ell$-vector spaces $V \cong \mathbb{Q}_\ell[t]/f(t)\mathbb{Q}_\ell[t]$ such that E becomes multiplication by t in $\mathbb{Q}_\ell[t]/f(t)\mathbb{Q}_\ell[t]$. Let x be an image of t in $\mathbb{Q}_\ell[t]/f(t)\mathbb{Q}_\ell[t]$, and let $R = \mathbb{Z}_\ell[x]$ be the $\mathbb{Z}_\ell$-subalgebra of $\mathbb{Q}_\ell[t]/f(t)\mathbb{Q}_\ell[t]$ generated by x. Then R is a $\mathbb{Z}_\ell$-lattice in $\mathbb{Q}_\ell[t]/f(t)\mathbb{Q}_\ell[t]$; in particular, the natural map $R \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to \mathbb{Q}_\ell[t]/f(t)\mathbb{Q}_\ell[t]$ is an isomorphism of $\mathbb{Q}_\ell$-vector spaces. We have to find an R-submodule $T \subset R \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ of $\mathbb{Z}_\ell$-rank d such that x acts on T with Hodge polygon $\mathrm{Hp}_\ell(G, d)$.

Suppose $G = \oplus_{i=1}^d \mathbb{Z}/\ell^{m_i}\mathbb{Z}$, where $m_1 \le m_2 \le \cdots \le m_d$ (recall that m_i may be zero for some i). First we construct certain elements $v_0, \ldots, v_d \in R \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Let us put $M(0) = 0$, and $M(s) = \sum_{i=1}^s m_i$

for $s \geq 1$. Let $f(t) = \sum_{i=0}^{d} a_i t^i$. We let

$$v_s = \frac{a_d x^s + \sum_{j=1}^{s} a_{d-j} x^{s-j}}{\ell^{M(s)}},$$

in particular, $v_d = f(x)/\ell^m = 0$. Second, we define T as the $\mathbb{Z}_\ell$-submodule of $R \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ generated by d elements $v_0, \ldots, v_{d-1}$. Note that $v_0, v_2, \ldots, v_{d-1}$ have different degrees viewed as polynomials in x (and all degrees are strictly less than d), and hence form a basis of T over $\mathbb{Z}_\ell$. In particular, T is a free $\mathbb{Z}_\ell$-module of rank d. Notice that $v_0 = a_d = (-1)^d$. In particular, T contains $\mathbb{Z}_\ell \cdot 1$.

Now we prove that T is an R-submodule of $R \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. The point $(d - s, M(s))$ is a vertex of $\mathrm{Hp}_\ell(G, d)$. By assumption $\mathrm{Np}_\ell(f)$ lies on or above $\mathrm{Hp}_\ell(G, d)$; thus $(d - s, M(s))$ is not higher than $\mathrm{Np}_\ell(f)$. It follows that $\ell^{M(s)}$ divides a_{d-s}, and

$$u_s = \frac{a_{d-s}}{\ell^{M(s)}} \cdot 1 \in \mathbb{Z}_\ell \cdot 1 \subset T,$$

thus for $s \geq 1$

$$xv_{s-1} = \ell^{m_s}(v_s - u_s) \in \ell^{m_s} T. \tag{2}$$

This proves that $xT \subset T$.

Recall that, $v_0 = a_d = (-1)^d$; clearly $m = \mathrm{ord}_\ell(\det E) = \mathrm{ord}_\ell(a_0)$; thus $u_d = a_0/\ell^m$ is invertible in $\mathbb{Z}_\ell$, and

$$v_1 - u_1, \ldots, v_d - u_d = -u_d$$

is a basis of the free $\mathbb{Z}_\ell$-module T. This gives us the natural surjective map $\mathbb{Z}^d \to T/xT$. It follows from (2) that this map factors through a surjective map $G \to T/xT$. We conclude that $T/xT \cong G$, since orders of both groups are equal to ℓ^m. □

Proof of theorem 1. The "if" part follows from Corollary 2. Let us prove the "only if" part.

For a given prime number ℓ and an abelian variety A' isogenius to A we let $V = V_\ell(A')$ and $E = 1 - F$. If $\ell \neq p$, we let $f(t) = f_A(1 - t)$, and if $\ell = p$, we let $f(t) = f_1(1 - t)$. By Theorem 3 there exists an E-invarint lattice T in V such that $T/ET \cong G_\ell$. Clearly, T is F-invariant, and by Lemma 1 there exists an abelian variety B' and an ℓ-isogeny $B' \to A'$ such that $T_\ell(B') \cong T$. By Proposition 1 we have $G_\ell \cong B'(k)_\ell$.

Let $\ell_1, \ldots, \ell_s$ be the set of prime divisors of $f_A(1)$. It follows that there exists a sequence of isogenies

$$B = B_s \xrightarrow{\varphi_s} B_{s-1} \to \ldots \xrightarrow{\varphi_2} B_1 \xrightarrow{\varphi_1} A$$

such that $\varphi_i : B_i \to B_{i-1}$ is an ℓ_i-isogeny and

$$B_i(k)_{\ell_i} \cong G_{\ell_i}.$$

Since φ_i is an ℓ_i-isogeny, $T_\ell(B_i) \cong T_\ell(B_{i-1})$ for any $\ell \neq \ell_i$. Thus $B(k) \cong G$. □

4. Noncommutative endomorphism algebras

If $\mathrm{End}\, A$ is not commutative then we may apply the following construction. Let $f_A = \prod_{j=1}^{s} f_j$, where all f_i are polynomials with integer coefficients, and f_j divides f_{j-1}. Suppose f_j has no multiple roots for $1 \leq j \leq s$. Let G_j be a family of abelian ℓ-groups for $1 \leq j \leq s$ such that $\mathrm{Np}_\ell(f_j(1 - t))$ lies on or above $\mathrm{Hp}_\ell(G_j, \deg f_j)$. By Theorem 3 and Lemma 1, we can construct modules T_j and an abelian variety B with Tate module $T_\ell(B) = \oplus T_j$ such that $B(k)_\ell \cong \oplus G_j$. We have the following conjecture.

Conjecture 1. *Let $f_A = \prod_{j=1}^{s} f_j$, where f_j divides f_{j-1}, and suppose that f_j has no multiple roots for $1 \le j \le s$. If $\deg f_j \le 2$ for all j, then $A(k)_\ell \cong \oplus G_j$, where G_j are ℓ-primary abelian groups such that $\mathrm{Np}_\ell(f_j(1-t))$ lies on or above $\mathrm{Hp}_\ell(G_j, \deg f_j)$ for all $1 \le j \le s$.*

This conjecture is proved in [Xi94] for simple abelian surfaces. However, there is an example of the group of points on an abelian variety A with $\deg f_1 = 3$, and $\deg f_2 = 1$ such that this group is not a direct sum of two groups G_1 and G_2 such that $\mathrm{Np}_\ell(f_j(1-t))$ lies on or above $\mathrm{Hp}_\ell(G_j, \deg f_j)$ for $j = 1, 2$. Let us consider the Weil polynomial $f(t) = (t^2 - 2t + 9)(t + 3)^2$, and let A be an abelian surface such that $f_A = f$. Then $f(1-t) = (t^2 + 8)(t-4)^2$. Let v_1, v_2, v_3, v_4 be a basis of $V_2(A)$ such that $(1-F)v_1 = 2v_2$, $(1-F)v_2 = -4v_1$, $(1-F)v_3 = 4v_3$, and $(1-F)v_4 = 4v_4$. Let T be the $\mathbb{Z}_2$-submodule of $V_2(A)$ generated by $u_1 = v_1 + v_3, u_2 = -4v_2 + 4v_3, u_3 = v_2 + v_4$, and $u_4 = 4v_1 + 2v_4$. By Lemma 1 there exists an abelian surface B such that $T_2(B) \cong T$. We claim that

$$B(\mathbb{F}_9)_2 \cong \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/16\mathbb{Z}.$$

Indeed,

$$\begin{gathered}(1-F)u_1 = 4u_1 + 2u_3 - u_4 \in T \\ (1-F)u_2 = 16u_3 \in T \\ (1-F)u_3 = 4u_1 - u_2 + 4u_3 \in T \\ (1-F)u_4 = 8u_1 \in T.\end{gathered}$$

Theorem 1 and Conjecture 1 allows one to classify the groups of points on simple abelian surfaces. However the author does not know even a conjectural classification of groups of points on nonsimple surfaces.

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